A Dynamic One-Dimensional Interface Interacting with a Wall

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We study a symmetric randomly moving line interacting by exclusion with a wall. We show that the expectation of the position of the line at the origin when it starts attached to the wall satisfies the following bounds:

$$c_1 t^{1/4} \leqslant \mathbb{E}\xi_t(0) \leqslant c_2 t^{1/4} \log t$$

The result is obtained by comparison with a "free" process, a random line that has the same behavior but does not see the wall. The free process is isomorphic to the symmetric nearest neighbor one-dimensional simple exclusion process. The height at the origin in the interface model corresponds to the integrated flux of particles through the origin in the simple exclusion process. We compute explicitly the asymptotic variance of the flux and show that the probability that this flux exceeds $K t^{1/4} \log t$ is bounded above by const. t^{2-K}. We have also performed numerical simulations, which indicate $\mathbb{E}\xi_t(0)^2 \sim t^{1/2} \log t$ as $t \to \infty$.

1. INTRODUCTION

We consider a process ξ_t on

$$\mathscr{X} = \{\xi \in \mathbb{N}^{\mathbb{Z}} : |\xi(x) - \xi(x+1)| = 1, \ \xi(0) \text{ even}\}$$

the space of trajectories of nearest neighbor random walks that stay non negative and such that at even "times" the walk visits even integers.

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The generator of the process is given by

$$\mathscr{L}f(\xi) = \tfrac{1}{2} \sum_x \mathbf{1}\{\xi + \Delta\xi(x)\, \delta_x \geqslant 0\}[f(\xi + \Delta\xi(x)\, \delta_x) - f(\xi)] \qquad (1.1)$$

where δ_x is the infinite vector having 1 in the xth coordinate and zero on the others. The sum $\xi + a\delta_x$ is understood coordinatewise. The discrete Laplacian Δ is defined by

$$\Delta\xi(x) := \xi(x+1) - 2\xi(x) + \xi(x-1) \qquad (1.2)$$

In words we can describe the dynamics as follows. The discrete Laplacian assumes only three values, -2, 0 and 2. When the Laplacian is zero, the interface does not move. When it is -2 or 2, at rate $\tfrac{1}{2}$ it makes a jump of length 2 in the same direction as the sign of the Laplacian. Over this motion we impose a restriction to keep the process in $\mathscr{X}$: the interface cannot be negative, so we simply prohibit the jumps which violate the restriction. This is the meaning of the indicator function $\mathbf{1}\{\xi + \Delta\xi(x)\, \delta_x \geqslant 0\}$ in the generator. We can think the prohibition of becoming negative as the interaction by exclusion of the interface with a wall at -1. For shortness we call ξ_t the *wall process*.

Our main result is the following

Theorem 1.3. Let ξ_t be the process with generator (1.1) and initial *flat configuration*:

$$\xi_0(x) := x \,(\mathrm{mod}\; 2) \qquad (1.4)$$

Then there exist positive constants c_1 and c_2 such that

$$c_1 t^{1/4} \leqslant \mathbb{E}\xi_t(0) \leqslant c_2 t^{1/4} \log t \qquad (1.5)$$

for sufficiently large t.

Theorem 1.3 catches the effect of the "entropic repulsion" in a stochastically moving interface interacting with a wall by exclusion.

The line induced in $\mathbb{R}^2$ by joining $(x, \xi_t(x))$ to $(x+1, \xi_t(x+1))$ for all $x \in \mathbb{Z}$ has the same behavior as the interface between -1's and 1's in a zero-temperature two-dimensional nearest-neighbors Ising model with a positive external field in the semiplane below the diagonal $x = y$ and with initial condition "all plus" below the diagonal and "all minus" above it. See Section 5 for details.

The equilibrium statistical mechanics of this model is well known. If one considers the generator $\mathscr{L}$ restricted to the box $[-L, L]$ with boundary conditions $\xi_t(-L) = \xi_t(L) = 0$, the invariant distribution is the uniform

distribution in the set of nearest neighbors random walk trajectories starting at time $-L$ at the origin, finishing at time L at the origin and being non negative for all intermediate times. Actually the uniform measure is even reversible for the process. But the uniform measure in this set corresponds to the law of a symmetric nearest neighbors random walk X_i conditioned to the set $\{X_{-L} = X_L = 0, X_i \geqslant 0, i \in [-L, L]\}$. Hence, the typical height of a configuration ξ with the invariant law in the bulk of the box is

$$\xi([rL]) \sim O(\sqrt{rL})$$

More precisely, the normalized process process $(L^{1/2}\xi([Lr]), r \in [-1, 1])$ converges as $L \to \infty$ to Brownian excursion on $[-1, 1]$; see Theorem 2.6 of Kaigh.[1]

Many papers deal with the problem of entropic repulsion in Equilibrium Statistical Mechanics. The role of the entropic repulsion in the Gaussian free field was studied by Lebowitz and Maes,[2] Bolthausen et al.,[3] Deuschel[4] and Deuschel and Giacomin.[5]

Entropic repulsion for Ising, SOS and related models was discussed in Bricmont et al.,[6] Bricmont,[7] Holický and Zahradník,[8] Cesi and Martinelli,[9] Lebowitz et al.,[10] Dinaburg and Mazel[11] and Ferrari and Martínez.[12]

The exponent $1/4$ for dynamic entropic repulsion was predicted by Lipowsky[13] using scaling arguments. This exponent was then found numerically by Mon et al.,[14] Albano et al.,[15] see Binder,[16] De Coninck et al.[17] It has also been observed in real experiments by Bartelt et al.[18] Further theoretical investigations of dynamics of lines, in relation to experiments can be found in Blagojevic and Duxbury.[19]

Dynamic entropic repulsion for a line of finite extension L when t, $L \to \infty$ strongly depends on the ratio t/L^2. The present paper deals with $L = \infty$ (analytical) or $t/L^2 \to 0$ (numerical). The case $t/L^2 = \mathcal{O}(1)$ has been studied by Funaki and Olla.[20]

The exponent $1/4$ also applies to the growth of fluctuations of an initially straight interface not interacting with the wall (see (2.59) later). For the Gaussian case, explicit computations were made by Abraham and Upton,[21] Abraham et al.[22] It was observed numerically in the two-dimensional Ising model by Stauffer and Landau.[23]

The strategy to show Theorem 1.3 is to compare the wall process ξ_t with a *free* process ζ_t having the same local dynamics as ξ_t but not interacting with the wall. The free process lives in

$$\mathscr{X}_0 = \{\zeta \in \mathbb{Z}^{\mathbb{Z}} : |\zeta(x) - \zeta(x+1)| = 1, \zeta(0) = \text{even}\}$$

and its generator is

$$\mathscr{L}_0 f(\zeta) = \tfrac{1}{2} \sum_x \left[f(\zeta + \Delta\zeta(x)\,\delta_x) - f(\zeta) \right] \tag{1.6}$$

In the next section we prove that with flat initial condition the variance of the height at the origin for the free process behaves as $t^{1/2}$:

$$\lim_{t \to \infty} t^{-1/2} \mathbb{V}\zeta_t(0) = \frac{1}{\sqrt{\pi}} \tag{1.7}$$

We then couple the wall process and the free process in such a way that

$$\zeta_t(x) \leqslant \xi_t(x) \tag{1.8}$$

for all x and t. The free process has enough symmetry and, properly rescaled, has uniformly bounded in time exponential moments. With these, (1.7) and (1.8), we get the lower bound in (1.5).

The idea for the upperbound is to consider a family of free processes with initial condition depending on t:

$$\zeta_0^{a_t}(x) = \zeta_0(x) + a_t$$

(a flat interface of height a_t). Then we fix $a_t = ct^{1/4} \log t$, the constant c to be determined later and exhibit a coupling under which

$$\xi_s(0) \leqslant \zeta_s^{a_t}(0) \tag{1.9}$$

for all $s \leqslant t$ with large probability. Combined with (1.7), inequality (1.9) is the key for the upperbound in (1.5). The existence of exponential moments (mentioned above) yields the moderate deviations result needed here.

The control of the fluctuations of the position at the origin of the free process is obtained by an isomorphism between the free process and the one-dimensional symmetric nearest-neighbor simple exclusion process. Under this map, $\zeta_t(0) = 2J_t$, where J_t is the integrated flux of particles at the origin for the exclusion process. We compute explicitly the asymptotic variance of the integrated flux for the flat initial condition in Theorem 2.17 later and obtain

$$\lim_{t \to \infty} \frac{\mathbb{V}J_t}{\sqrt{t}} = \frac{1}{4\sqrt{\pi}} \tag{1.10}$$

De Masi and Ferrari[24] proved that the asymptotic variance of the integrated flux when the initial configuration is distributed according to a product measure with density $1/2$ is given by

$$\lim_{t \to \infty} \frac{\mathbb{V} J_t}{\sqrt{t}} = \frac{1}{2\sqrt{2\pi}} \tag{1.11}$$

which is strictly bigger than (1.10). When the initial density is ρ, the asymptotic variance is given by $\rho(1-\rho)\sqrt{2/\pi}$. The method to show (1.10) and (1.11) is based on duality and comparison with systems of independent particles and it is inspired by Arratia,[25] who used these tools to compute the variance of a tagged particle for the process starting with an (invariant) product measure. However a modification of Arratia's proof is needed in (1.10) due to the deterministic character of the initial configuration.

The study of the flux in the simple exclusion process is done in Section 2. In Section 3 we prove Theorem 1.3. Section 4 is devoted to numerical simulations. Section 5 shows the equivalence to the dynamics of a particular zero temperature Ising model interface.

2. THE FREE PROCESS AND SIMPLE EXCLUSION

The *simple exclusion process* lives in $\{0, 1\}^{\mathbb{Z}}$ and its generator is

$$\mathscr{L}^{ex} f(\eta) = \tfrac{1}{2} \sum_{x \in \mathbb{Z}} [f(\eta^{x, x+1}) - f(\eta)] \tag{2.1}$$

where

$$\eta^{x, x+1}(y) = \begin{cases} \eta(y) & \text{if } y \neq x, x+1 \\ \eta(x+1) & \text{if } y = x \\ \eta(x) & \text{if } y = x+1 \end{cases} \tag{2.2}$$

It is convenient to construct the processes using the Harris graphical construction.

Harris Graphical Construction. Let $(N_t(x) : x \in \mathbb{Z})$ be a family of independent Poisson processes of rate $\tfrac{1}{2}$. For each x, $N_t(x)$ counts the number of Poisson events associated to x in the time interval $[0, t]$. Denote $dN_t(x) = \mathbf{1}\{$there is a Poisson event associated to x at time $t\} = \lim_{h \to 0}(N_t(x) - N_{t-h}(x))$. Let η_t be the process defined by

$$d\eta_t(x) = (\eta_t(x-1) - \eta_t(x))\, dN_t(x) + (\eta_t(x+1) - \eta_t(x))\, dN_t(x+1) \tag{2.3}$$

The process is well defined because for each finite time t the value of the process in a finite box can be determined by looking at only a finite but random number of Poisson events and initial values. See for instance Arratia.[25] In words, the motion can be described as follows. The Poisson marks of $N_t(x)$ are associated to the bond $(x, x+1)$ and each time a Poisson mark occurs, the contents of the associated bond are interchanged. It is immediate to show that this process has generator (2.1).

   Stirring Particles.   To introduce the notion of duality and to deal with the flux of particles it is convenient to follow the "stirring particles" as defined by Arratia.[25] Let X_t^x be the position at time t determined by $X_0^x = x$ and the equations

$$dX_t^x = dN_t(X_t^x) - dN_t(X_t^x - 1) \tag{2.4}$$

So that, each time a Poisson mark associated to one of the neighboring bonds of a particle occurs, the particle jumps across the bond. Of course, if both extremes of a bond are occupied, the particles jump simultaneously, respecting the exclusion condition "at most one particle per site." For each $t \geqslant 0$ the (random) map

$$x \mapsto X_t^x \tag{2.5}$$

is a bijection of $\mathbb{Z}$ in $\mathbb{Z}$. The (marginal) law of X_t^x is a symmetric nearest neighbor random walk starting at x.

   Duality.   Let $y \mapsto D_t^y$ be the inverse map defined by $x = D_t^y$ if and only if $y = X_t^x$. The following "duality formula" holds immediately

$$\eta_t(y) = \eta_0(D_t^y) \tag{2.6}$$

So,

$$\prod_{y \in A} \eta_t(y) = \prod_{y \in A} \eta_0(D_t^y) \tag{2.7}$$

Notice that for a finite set of sites A, $\{D_t^y : y \in A\}$ has the same one-time marginal as a simple exclusion process with initial condition A (here we are identifying the configuration η with the set $\{x : \eta(x) = 1\}$). When $A = \{y\}$ (contains only one site), the one-time marginal D_t^y has the same law as X_t^y for all $t \geqslant 0$.

Integration by Parts Formula. Consider (Y_t^i, Z_t^j) independent random walks with the same marginals as the stirring process (X_t^i, X_t^j). The generator of the process (Y_t^i, Z_t^j) is the following:

$$Uf(i, j) = \tfrac{1}{2} \sum_{e \in \mathbb{Z}^2 : |e| = 1} [f((i, j) + e) - f(i, j)] \tag{2.8}$$

and the generator of the process X_t is

$$Vf(i, j) = \begin{cases} \tfrac{1}{2} \sum_{e \in \mathbb{Z}^2 : |e| = 1} [f((i, j) + e) - f(i, j)] & \text{if} \quad i - j > 1 \\ \tfrac{1}{2} f(j, i) + \tfrac{1}{2} f(i+1, j) + \tfrac{1}{2} f(i, j-1) - \tfrac{3}{2} f(i, j) & \text{if} \quad i - j = -1 \\ \tfrac{1}{2} f(j, i) + \tfrac{1}{2} f(i-1, j) + \tfrac{1}{2} f(i, j+1) - \tfrac{3}{2} f(i, j) & \text{if} \quad i - j = 1 \end{cases} \tag{2.9}$$

Hence, for $i \neq j$,

$$Uf(i, j) - Vf(i, j) = -\tfrac{1}{2} \mathbf{1}\{|i - j| = 1\}(f(i, j) + f(j, i) - f(i, i) - f(j, j)) \tag{2.10}$$

Let U_t and V_t be the semigroups generated by U and V respectively. Let $f : \mathbb{Z}^2 \to \mathbb{R}$. Then

$$\mathbb{E}f(X_t^i, X_t^j) - \mathbb{E}f(Y_t^i, Z_t^j) = [V_t - U_t] f(i, j) = \int_0^t V_s [U - V] U_{t-s} f(i, j) \, ds \tag{2.11}$$

where the last identity is the integration by parts formula (see Liggett[26] Proposition 8.1.7). Now, using (2.10) to compute (2.11) we get for $i \neq j$:

$$\mathbb{E}f(X_t^i, X_t^j) - \mathbb{E}f(Y_t^i, Z_t^j)$$
$$= -\tfrac{1}{2} \int_0^t ds \, \mathbb{E}(\mathbf{1}\{|X_s^i - X_s^j| = 1\}[f(Y_t^{X_{t-s}^i}, Z_t^{X_{t-s}^j}) + f(Y_t^{X_{t-s}^j}, Z_t^{X_{t-s}^i})$$
$$- f(Y_t^{X_{t-s}^i}, Z_t^{X_{t-s}^i}) - f(Y_t^{X_{t-s}^j}, Z_t^{X_{t-s}^j})]) \tag{2.12}$$

This identity will be used in the sequel.

Flux. Let J_t be the integrated flux of η particles through the point $-1/2$ in the exclusion process:

$$J_t := \sum_{x < 0} \eta_0(x) \, \mathbf{1}\{X_t^x \geqslant 0\} - \sum_{x \geqslant 0} \eta_0(x) \, \mathbf{1}\{X_t^x < 0\} \tag{2.13}$$

where X_t^x is the position at time t of the exclusion particle that at time zero was at position x.

Replacing (1.4) in (2.13), we write

$$J_t := \sum_{i<0} \mathbf{1}\{X_t^{2i} \geqslant 0\} - \sum_{i \geqslant 0} \mathbf{1}\{X_t^{2i} < 0\} \tag{2.14}$$

J_t is almost symmetric. Let

$$H_t := \sum_{i<0} \mathbf{1}\{X_t^{2i} \geqslant 0\}; \quad H_t' := \sum_{i \geqslant 0} \mathbf{1}\{X_t^{2i} < -1\}; \quad I_t := \sum_{i \geqslant 0} \mathbf{1}\{X_t^{2i} < 0\} \tag{2.15}$$

Then, clearly,

$$J_t = H_t - I_t; \qquad H_t \sim H_t'; \qquad |H_t' - I_t| \leqslant 1 \tag{2.16}$$

where $\sim$ means identity in distribution and is justified in this case by spatial and distributional symmetry.

Theorem 2.17. For the simple exclusion process with generator (2.1) and initial condition $\eta_0 = 1 - 2\xi_0$, as defined in (1.4),

$$\lim_{t \to \infty} \frac{\mathbb{V}J_t}{\sqrt{t}} = \frac{1}{4\sqrt{\pi}} \tag{2.18}$$

Proof. Working from (2.14), we get

$$\mathbb{E}(J_t)^2 = \sum_{i<0} \mathbb{P}(X_t^{2i} \geqslant 0) + \sum_{i \geqslant 0} \mathbb{P}(X_t^{2i} < 0) + 2 \sum_{i<0} \sum_{i<j<0} \mathbb{P}(X_t^{2i} \geqslant 0, X_t^{2j} \geqslant 0)$$

$$+ 2 \sum_{i \geqslant 0} \sum_{i>j \geqslant 0} \mathbb{P}(X_t^{2i} < 0, X_t^{2j} < 0) - 2 \sum_{i<0} \sum_{j \geqslant 0} \mathbb{P}(X_t^{2i} \geqslant 0, X_t^{2j} < 0) \tag{2.19}$$

$$(\mathbb{E}J_t)^2 = \sum_{i<0} \mathbb{P}^2(X_t^{2i} \geqslant 0) + \sum_{i \geqslant 0} \mathbb{P}^2(X_t^{2i} < 0)$$

$$+ 2 \sum_{i<0} \sum_{i<j<0} \mathbb{P}(X_t^{2i} \geqslant 0) \, \mathbb{P}(X_t^{2j} \geqslant 0)$$

$$+ 2 \sum_{i \geqslant 0} \sum_{i>j \geqslant 0} \mathbb{P}(X_t^{2i} < 0) \, \mathbb{P}(X_t^{2j} < 0)$$

$$- 2 \sum_{i<0} \sum_{j \geqslant 0} \mathbb{P}(X_t^{2i} \geqslant 0) \, \mathbb{P}(X_t^{2j} < 0) \tag{2.20}$$

Immediately we have:

$$\sum_{i<0} \mathbb{P}(X_t^{2i} \geq 0) + \sum_{i \geq 0} \mathbb{P}(X_t^{2i} < 0) = \sum_{i<0} \mathbb{P}(X_t^i \geq 0) \tag{2.21}$$

and analogously for the $\mathbb{P}^2$ terms in (2.20). Using

$$\mathbb{P}(A\, B) - \mathbb{P}(A)\, \mathbb{P}(B) = -(\mathbb{P}(A\, B^c) - \mathbb{P}(A)\, \mathbb{P}(B^c)) = \mathbb{P}(A^c\, B^c) - \mathbb{P}(A^c)\, \mathbb{P}(B^c) \tag{2.22}$$

we get

$$\mathbb{V}J_t = \mathscr{V}_t + \mathscr{E}_t \tag{2.23}$$

where

$$\mathscr{V}_t = \sum_{i<0} \mathbb{P}(X_t^i \geq 0) - \sum_{i<0} \mathbb{P}^2(X_t^i \geq 0) \tag{2.24}$$

and

$$\begin{aligned}
\mathscr{E}_t &= \left(\sum_{i,j<0,\, i \neq j} + \sum_{i,j \geq 0,\, i \neq j} + 2 \sum_{i<0,\, j \geq 0} \right) \\
&\quad \times (\mathbb{P}(X_t^{2i} \geq 0, X_t^{2j} \geq 0) - \mathbb{P}(X_t^{2i} \geq 0)\, \mathbb{P}(X_t^{2j} \geq 0)) \\
&= \sum_{i \neq j} (\mathbb{P}(X_t^{2i} \geq 0, X_t^{2j} \geq 0) - \mathbb{P}(X_t^{2i} \geq 0)\, \mathbb{P}(X_t^{2j} \geq 0)) \tag{2.25}
\end{aligned}$$

Since $\mathbb{P}(X_t^i \geq 0)\, \mathbb{P}(X_t^j \geq 0) = \mathbb{P}(Y_t^i \geq 0, Z_t^j \geq 0)$, we can use (2.12) with $f(i,j) = \mathbf{1}\{i \geq 0, j \geq 0\}$ to get

$$\begin{aligned}
&\mathbb{P}(X_t^i \geq 0, X_t^j \geq 0) - \mathbb{P}(X_t^i \geq 0)\, \mathbb{P}(X_t^j \geq 0) \\
&= -\tfrac{1}{2} \int_0^t \sum_y \mathbb{P}(\{X_s^i, X_s^j\} = \{y, y+1\}) (\mathbb{P}(Y_{t-s}^y \geq 0) - \mathbb{P}(Y_{t-s}^{y+1} \geq 0))^2 \, ds \tag{2.26}
\end{aligned}$$

See also Theorem 2 of Ferrari *et al.*[27] for a probabilistic proof of the previous identity. Translation invariance and self-duality of (X_s^i, X_s^j) implies that (2.26) equals

$$-\tfrac{1}{2} \int_0^t \sum_y \mathbb{P}(\{X_s^0, X_s^1\} = \{i-y, j-y\})\, \mathbb{P}^2(Y_{t-s}^0 = y) \, ds \tag{2.27}$$

From (2.25)–(2.27),

$$\mathcal{E}_t = -\tfrac{1}{2} \int_0^t \sum_y \mathbb{P}^2(Y_{t-s}^0 = y) \sum_{i \neq j} \mathbb{P}(\{X_s^0, X_s^1\} = \{2i - y, 2j - y\}) \, ds \quad (2.28)$$

Since $X_t^0 \neq X_t^1$,

$$\sum_{i \neq j} \mathbb{P}(\{X_s^0, X_s^1\} = \{2i - y, 2j - y\}) = 2\, \mathbb{P}(X_s^1 \ (\mathrm{mod}\ 2)$$

$$= X_s^0 \ (\mathrm{mod}\ 2) = y \ (\mathrm{mod}\ 2)) \quad (2.29)$$

Let $A_y = \{(i, j) \in \mathbb{Z}^2 : i \ (\mathrm{mod}\ 2) = j \ (\mathrm{mod}\ 2) = y \ (\mathrm{mod}\ 2)\}$. We show below that

$$\lim_{s \to \infty} \mathbb{P}((X_s^0, X_s^1) \in A_y) = 1/4 \quad (2.30)$$

uniformly in y. Hence,

$$\lim_{t \to \infty} t^{-1/2} \mathcal{E}_t = -\tfrac{1}{4} \lim_{t \to \infty} t^{-1/2} \int_0^t \ \sum_y \mathbb{P}^2(Y_s^0 = y) \, ds \quad (2.31)$$

Let Z_t^0 be an independent copy of Y_t^0. Since $\sum_y \mathbb{P}^2(Y_s^0 = y) = \mathbb{P}(Y_t^0 - Z_t^0 = 0)$, changing variables the above limit equals

$$\lim_{t \to \infty} \int_0^1 (st)^{1/2}\, \mathbb{P}(Y_{st}^0 - Z_{st}^0 = 0) \frac{ds}{\sqrt{s}} = \int_0^1 \lim_{t \to \infty} (st)^{1/2}\, \mathbb{P}(Y_{st}^0 - Z_{st}^0 = 0) \frac{ds}{\sqrt{s}} \quad (2.32)$$

where the interchange of the limit and the integral are guaranteed by the local central limit theorem for $(Y_t^0 - Z_t^0)$, which is a symmetric random walk of rate 2. This also implies that (2.32) equals

$$\int_0^1 \frac{1}{\sqrt{2\pi}\,\sqrt{2}} \frac{ds}{\sqrt{s}} = \frac{1}{\sqrt{\pi}} \quad (2.33)$$

We conclude that

$$\lim_{t \to \infty} t^{-1/2} \mathcal{E}_t = -\frac{1}{4\sqrt{\pi}} \quad (2.34)$$

To compute $\mathscr{V}_t$ notice that

$$\sum_{i<0} \mathbb{P}(X_t^i \geqslant 0) = \sum_{i>0} \mathbb{P}(X_t^0 \geqslant i) = \mathbb{E}((X_t^0)^+); \tag{2.35}$$

$$\sum_{i<0} \mathbb{P}^2(X_t^i \geqslant 0) = \sum_{i>0} \mathbb{P}^2(X_t^0 \geqslant i) = \sum_{i>0} \mathbb{P}(Y_t^0 \wedge Z_t^0 \geqslant i) = \mathbb{E}[(Y_t^0 \wedge Z_t^0)^+] \tag{2.36}$$

Thus

$$\lim_{t \to \infty} t^{-1/2} \mathscr{V}_t = \mathbb{E}(X^+) - \mathbb{E}[(X \wedge X')^+] = \frac{1}{2\sqrt{\pi}} \tag{2.37}$$

where X and X' are i.i.d. standard normals.

Finally, substituting (2.34) and (2.37) in (2.23) we get (2.18). ∎

Proof of 2.30. The continuous time Markov chain $(Y_t^0 \pmod 2), Y_t^1 \pmod 2)$ converges exponentially fast to the uniform distribution in $\{(0, 1), (1, 0), (0, 0), (1, 1)\}$. This implies that there exist positive constants C_1, C_2 such that

$$|\mathbb{P}((Y_s^i, Z_s^j) \in A_y) - 1/4| \leqslant C_1 e^{-C_2 t} \tag{2.38}$$

uniformly in i, j, y. Writing $f_y(i, j) := \mathbf{1}\{(i, j) \in A_y\}$ and using (2.12) we get

$$|\mathbb{P}((X_t^0, X_t^1) \in A_y) - \mathbb{P}((Y_t^0, Z_t^1) \in A_y)|$$

$$\leqslant \tfrac{1}{2} \int_0^t ds \, \mathbb{E}(\mathbf{1}\{|X_s^0 - X_s^1| = 1\}[f_y(Y_t^{X_{t-s}^0}, Z_t^{X_{t-s}^1}) + f_y(Y_t^{X_{t-s}^1}, Z_t^{X_{t-s}^0})$$

$$- f_y(Y_t^{X_{t-s}^0}, Z_t^{X_{t-s}^0}) - f_y(Y_t^{X_{t-s}^1}, Z_t^{X_{t-s}^1})]) \tag{2.39}$$

$$\leqslant 2 \int_0^t ds \, \mathbb{P}(|X_s^1 - X_s^0| = 1) \, C_1 \, e^{-C_2(t-s)} \tag{2.40}$$

(using (2.38)). Now $|X_s^1 - X_s^0|$ is a Markov chain in $\{1, 2,...\}$ with rates $p(1, 2) = p(x, x+1) = p(x, x-1) = 1/2$, $x > 1$. It can be easily coupled to a Markov chain in $\{0, 1, 2,...\}$ starting in 0, say R_t^0, with rates $p(0, 1) = 1$, $p(x, x+1) = p(x, x-1) = 1/2$, $x > 0$ in such a way that $|X_t^1 - X_t^0| \geqslant R_s^0$ for all s. Since R_t^0 is a simple symmetric random walk reflected at the origin, we get that $\lim_{s \to \infty} \mathbb{P}(|X_s^1 - X_s^0| = 1) \leqslant \lim_{s \to \infty} \mathbb{P}(|R_s^0| \leqslant 1) = 0$ and thus, from (2.40) and dominated convergence (after a change of variables $s \to t-s'$), it follows that $\lim_{t \to \infty} \mathbb{P}((X_s^0, X_s^1) \in A_y) = \lim_{t \to \infty} \mathbb{P}((Y_s^0, Z_s^1) \in A_y) = 1/4$. ∎

Lemma 2.41. Let H_t be as in (2.16) and $\tilde{H}_t = t^{-1/4}[H_t - \mathbb{E}(H_t)]$. Then for all $\lambda \in \mathbb{R}$

$$\limsup_{t \to \infty} \mathbb{E}(e^{\lambda \tilde{H}_t}) \leqslant e^{s\lambda^2/2} \tag{2.42}$$

where $s = 1/\sqrt{2\pi}$.

Proof.

$$\mathbb{E}(e^{\lambda \tilde{H}_t}) = \frac{\mathbb{E}[\exp{(\lambda t^{-1/4} \sum_{i<0} \mathbf{1}\{X_t^{2i} \geqslant 0\})}]}{\exp{(\lambda t^{-1/4} \sum_{i<0} \mathbb{P}(X_t^{2i} \geqslant 0))}} \tag{2.43}$$

We will show that the quotient in (2.43) is bounded above by a constant. For that, we need to evaluate the expected value in that equation. Let $\lambda \geqslant 0$. We will argue below that

$$\mathbb{E}\left[\exp\left(\lambda t^{-1/4} \sum_{i<0} \mathbf{1}\{X_t^{2i} \geqslant 0\} \right) \right] \leqslant \prod_{i<0} \mathbb{E}[\exp{(\lambda t^{-1/4}\mathbf{1}\{X_t^{2i} \geqslant 0\})}] \tag{2.44}$$

The last expectation equals

$$1 + [\exp(\lambda t^{-1/4}) - 1]\, \mathbb{P}(X_t^{2i} \geqslant 0)$$
$$= 1 + [\lambda t^{-1/4} + (\lambda^2 t^{-1/2}/2) + o(t^{-1/2})]\, \mathbb{P}(X_t^{2i} \geqslant 0) \tag{2.45}$$

for all t large enough. The last expression is bounded above by

$$\exp\{[\lambda t^{-1/4} + (\lambda^2 t^{-1/2}/2) + o(t^{-1/2})]\, \mathbb{P}(X_t^{2i} \geqslant 0)\} \tag{2.46}$$

Substituting into the right hand side of (2.44), we get

$$\exp\left\{ [(\lambda^2 t^{-1/2}/2) + o(t^{-1/2})] \sum_{i<0} \mathbb{P}(X_t^{2i} \geqslant 0) \right\} \tag{2.47}$$

as an upper bound for the quotient in (2.43). It is not difficult to see that the expression on the exponent in (2.47) converges to $e^{s\lambda^2/2}$.

To finish the argument for $\lambda \geqslant 0$, we have to justify the inequality (2.44). That follows from taking limits as $M \to -\infty$ (and using monotone convergence) on the respective inequalities gotten by replacing the infinite sums by $\sum_{M<i<0}$. These are justified by the fact that the functions $\exp(t^{-1/4} \sum_{M<i<0} \mathbf{1}\{X_t^{2i} > 0\})$ are bounded, symmetric and positive definite for all $M < 0$. The inequalities then follow from Proposition 1.7, Chapter VIII of Liggett.[26]

For the case $\lambda < 0$, we use the identity

$$\sum_{i<0} \mathbf{1}\{X_t^{2i} \geqslant 0\} - \sum_{i<0} \mathbb{P}(X_t^{2i} \geqslant 0) = -\left[\sum_{i<0} \mathbf{1}\{X_t^{2i} < 0\} - \sum_{i<0} \mathbb{P}(X_t^{2i} < 0) \right] \tag{2.48}$$

and a similar argument as above. ∎

Lemma 2.49. Let η_0 be given by the flat condition as in Theorem 2.17. Then

$$\sup_{t \geqslant 0} \mathbb{E}(e^{|J_t/t^{1/4}|}) < \infty \tag{2.50}$$

Furthermore for all $K > 0$ and all t large enough

$$\mathbb{P}(|J_t| > Kt^{1/4} \log t) \leqslant ct^{-K} \tag{2.51}$$

where c is a constant.

Proof. The bound (2.50) follows straightforwardly from Lemma 2.41 and relations (2.16).

From the relations (2.16), to show (2.51) it is enough to prove the result with $|H_t - \mathbb{E}(H_t)|$ replacing $|J_t|$ in (2.51) (the constant of course does not need to be the same). We have

$$\mathbb{P}(|H_t| > Kt^{1/4} \log t) = \mathbb{P}(|\tilde{H}_t| > \log t^K) \leqslant c't^{-K} \tag{2.52}$$

where the last inequality follows from the exponential Markov inequality and $c' = \sup_{t \geqslant 0} \mathbb{E}(e^{|\tilde{H}_t|})$ is finite by (2.50). ∎

Graphical Construction of Free Process. Let ζ_t be the process defined by

$$d\zeta_t(x) = \Delta\zeta_t(x) \, dN_t(x) \tag{2.53}$$

where the discrete Laplacian Δ was defined in (1.2). In words, each time a Poisson mark of the process $N_t(x)$ occurs, the height at x at time t decreases or increases two unities, according to the value of the Laplacian at this point at this time; if the Laplacian vanishes, no jump occurs. This process has generator (1.6).

Lemma 2.54. Let $\eta_0(x) = \zeta_0(x+1) - \zeta_0(x)$. Then

$$\eta_t(x) = \zeta_t(x+1) - \zeta_t(x) \tag{2.55}$$

where the processes ζ_t and η_t are defined by (2.3) and (2.53) and have initial conditions η_0 and ζ_0 respectively. Furthermore,

$$\zeta_t(0) - \zeta_0(0) = 2J_t \tag{2.56}$$

Proof. Notice that from (2.55),

$$\Delta\zeta_t(0) = 2(\eta_t(-1) - \eta_t(0)) \tag{2.57}$$

Assume that there is a mark of the process $N_t(-1)$ at time t. Then (2.57), (2.3) and (2.53) imply that if $\eta_t(-1) - \eta_t(0) = 0$ no changes occur neither for $\eta_t(-1)$, $\eta_t(0)$ nor for $\zeta_t(0)$; if $\eta_t(-1) - \eta_t(0) = 1$, an exclusion particle jumps from -1 to 0 and the free process at the origin jumps two units up; if $\eta_t(-1) - \eta_t(0) = -1$, an exclusion particle jumps from 0 to -1 and the free process at the origin jumps two units down. Identity (2.56) follows from (2.55). ∎

Lemma 2.58. Let ζ_t be the free process with flat initial condition (1.4). Then

$$\lim_{t \to \infty} \frac{\mathbb{V}\zeta_t(0)}{\sqrt{t}} = \frac{1}{\sqrt{\pi}}; \tag{2.59}$$

$$\sup_{t \geqslant 0} \mathbb{E}(e^{|\zeta_t(0)/t^{1/4}|}) < \infty \tag{2.60}$$

and for all $K > 0$ and all t large enough

$$\mathbb{P}(|\zeta_t(0)| > Kt^{1/4} \log t) \leqslant ct^{-K} \tag{2.61}$$

Proof. It follows from identity (2.56), the limit (2.18) and the bounds (2.50) and (2.51) ∎

3. COUPLING THE WALL AND THE FREE PROCESSES

We construct graphically the wall process which simultaneously provides another graphical construction for the free process. Under this construction the wall process dominates the free one. We consider two independent families of Poisson processes with the same law as $N_t(x)$ called $N_t^+(x)$ and $N_t^-(x)$, to be used for upwards and downwards jumps, respectively. The process satisfying the equations

$$\begin{aligned}
d\xi_t(x) = {} & \Delta\xi_t(x) \, \mathbf{1}\{\Delta\xi_t(x) > 0\} \, dN_t^+(x) \\
& + \Delta\xi_t(x) \, \mathbf{1}\{\Delta\xi_t(x) < 0, \, \xi_t(x) + \Delta\xi_t(x) \geqslant 0\} \, dN_t^-(x) \quad (3.1)
\end{aligned}$$

has generator (1.1). The process ζ_t satisfying

$$d\zeta_t(x) = \Delta\xi_t(x)\,\mathbf{1}\{\Delta\xi_t(x) > 0\}\,dN_t^+(x) + \Delta\xi_t(x)\,\mathbf{1}\{\Delta\xi_t(x) < 0\}\,dN_t^-(x) \tag{3.2}$$

has generator (1.6).

In words, when a time event of the process $N_t^+(x)$ occurs at time t, the process ξ_t at site x and time t jumps two units upwards if $\Delta\xi_t(x) > 0$. When a time event of the process $N_t^-(x)$ occurs at time t, the process ξ_t at site x and time t jumps two units downwards if $\Delta\xi_t(x) < 0$ *and* the wall condition $\xi_t(x) + \Delta\xi_t(x) \geqslant 0$ holds. The process satisfying (3.2) follows the same marks in the same manner but ignoring the wall condition. The difference with the process satisfying (2.53) is that in this case the Poisson events N_t are used for *both* upwards and downwards jumps; this construction is not attractive in the sense that it does not satisfy (3.4) below.

Let r be a non negative integer and ξ_t^r and ζ_t^r be the processes defined by (3.1) and (3.2) but with initial condition

$$\xi_0^r(x) = \zeta_0^r(x) = r + x \pmod 2 \tag{3.3}$$

Notice that ζ_t^0 and ζ_t as defined in (2.53) have the same law but are *different* processes. The processes ξ_t and ξ_t^r defined by (3.1) and the same initial condition satisfy

$$\xi_t(0) \leqslant \xi_t^r(0) \tag{3.4}$$

for all $r \geqslant 0$. This joint construction corresponds to what Liggett[26] calls *basic coupling*.

Lemma 3.5. There exists a constant $c > 0$ such that for any $K > 0$ and $t \geqslant 0$

$$\mathbb{P}(\xi_t(0) > 2Kt^{1/4}\log t) \leqslant ct^{2-K} \tag{3.6}$$

Proof. Let $a_t = 2Kt^{1/4}\log t$. Take $r \geqslant 0$ and write

$$\mathbb{P}(\xi_t(0) \geqslant a_t) \leqslant \mathbb{P}(\xi_t^r(0) \geqslant a_t)$$
$$= \mathbb{P}(\xi_t^r(0) \geqslant a_t,\, \xi_t^r(0) = \zeta_t^r(0)) + \mathbb{P}(\xi_t^r(0) \geqslant a_t,\, \xi_t^r(0) \neq \zeta_t^r(0))$$
$$\leqslant \mathbb{P}(\zeta_t^r(0) \geqslant a_t) + \mathbb{P}(\xi_t^r(0) \neq \zeta_t^r(0)) \tag{3.7}$$

The first term in (3.7) will be bounded using Corollary 2.58. To bound the second term notice that if the interacting process and the free process differ at the origin this is due to a collision of the interacting process with

the wall at some point x that separate the two processes at x at some time s; the discrepancy then propagates and arrives to zero by time t. We fix an $\alpha > 0$ and separate the discrepancies in two classes: those that come from the interval $[-\alpha t, \alpha t]$ and those that come from outside this interval. If in the time interval $[0, t]$ the free process does not touch the wall in the space interval $[-\alpha t, \alpha t]$, then the discrepancy must come from outside. Hence,

$$\{\xi_t^r(0) \neq \zeta_t^r(0)\} \subset \{\zeta_t^r(x) < 0 \text{ for some } s \in [0, t], x \in [-\alpha t, \alpha t]\}$$
$$\cup \{\text{a discrepancy from } [-\alpha t, \alpha t]^c \text{ reaches } 0 \text{ up to time } t\} \qquad (3.8)$$

Observe that the law of $\zeta_t^r(x) - r$ is the same as the law of $\zeta_t(0)$ and that $\mathbb{P}(\zeta_s(0) < -r) \leqslant \mathbb{P}(\zeta_s(0) > r)$ due to the initial condition being non-negative. Hence, fixing

$$r = a_t/2 \qquad (3.9)$$

the probability of the first event in the right hand side of (3.8) is bounded by

$$(2\alpha t + 1) \, \mathbb{P}(\zeta_s(0) > a_t/2 \text{ for some } s \in [0, t]) \qquad (3.10)$$

From (2.60) and the exponential Markov inequality, we have that $\sup_{s \leqslant t} \mathbb{P}(\zeta_s(0) > r) \leqslant ct^{-K}$ for some constant c, so we can bound (3.10) with

$$(2\alpha t + 1) \int_0^t ct^{-K} \, ds \leqslant (2\alpha + 1) \, ct^{2-K} \qquad (3.11)$$

To bound the probability of the second event in the right hand side of (3.8) notice that discrepancies cannot travel faster than N_t, a Poisson process of parameter 1. Hence

$$\mathbb{P}(\text{a discrepancy from } [-\alpha t, \alpha t]^c \text{ reaches } 0 \text{ up to time } t)$$
$$\leqslant 2\mathbb{P}(N_t > \alpha t) \leqslant 2 \, e^{-t(\alpha + 1 - e)} \qquad (3.12)$$

using the exponential Chebyshev inequality. Fixing $\alpha = 2$, and using the bounds (3.10) and (3.12), the probability of (3.8) is bounded by

$$4 \, ct^{2-K} + 2 \, e^{-t(3-e)} \leqslant c't^{2-K} \qquad (3.13)$$

for some constant c' and sufficiently large t. ∎

Proof of Theorem 1.3. It follows straightforwardly from (2.60) that $\tilde{\zeta}_t^2$ is uniformly integrable, where $\tilde{\zeta}_t = t^{-1/4}\zeta_t(0)$. This, together with (2.59), implies the lower bound in (1.5), as we will see now. Indeed, (1.8) implies that $t^{-1/4}\mathbb{E}\xi_t(0) \geqslant \mathbb{E}|\tilde{\zeta}_t|$. Now,

$$\mathbb{V}\tilde{\zeta}_t \leqslant \mathbb{E}\tilde{\zeta}_t^2 = \mathbb{E}(\tilde{\zeta}_t^2; \tilde{\zeta}_t^2 \leqslant M^2) + \mathbb{E}(\tilde{\zeta}_t^2; \tilde{\zeta}_t^2 > M^2) \leqslant M\mathbb{E}|\tilde{\zeta}_t| + \epsilon_M \qquad (3.14)$$

uniformly in t, where M is an arbitrary positive number and $\epsilon_M \to 0$ as $M \to \infty$. Thus $t^{-1/4}\mathbb{E}\xi_t(0) \geqslant (\mathbb{V}\tilde{\zeta}_t - \epsilon_M)/M$. We conclude that

$$\liminf_{t \to \infty} t^{-1/4}\mathbb{E}\xi_t(0) \geqslant \sup_{M>0} \left(\frac{1}{\sqrt{\pi}} - \epsilon_M\right)\bigg/M > 0 \qquad (3.15)$$

For the upperbound, we use 2.61 to obtain

$$\frac{\mathbb{E}\xi_t(0)}{t^{1/4}\log t} = \sum_{k \geqslant 0} \mathbb{P}(\xi_t(0) > kt^{1/4}\log t) \leqslant 4 + \sum_{k \geqslant 5} c't^{2-k/2} \leqslant c_2 < \infty \qquad (3.16)$$

for some constant $c_2 < \infty$. ∎

4. NUMERICAL SIMULATION

We have simulated the processes ζ_t and ξ_t numerically, using various pseudo-random number generators. The interface is of length L with periodic boundary conditions, so that the processes live in

$$\mathscr{X}_{0L} = \{\zeta \in \mathbb{Z}^{\mathbb{Z}/L\mathbb{Z}} : |\zeta(x) - \zeta(x+1)| = 1, \zeta(0) \text{ even}\} \qquad (4.1)$$

or

$$\mathscr{X}_L = \{\xi \in \mathbb{N}^{\mathbb{Z}/L\mathbb{Z}} : |\xi(x) - \xi(x+1)| = 1, \xi(0) \text{ even}\} \qquad (4.2)$$

Time is an integer multiple of $2L^{-1}$, i.e., $t \in (2L^{-1})\,\mathbb{N}$. For each time step, a site x is chosen randomly according to the uniform measure on $\mathbb{Z}/L\mathbb{Z}$, and the interface is updated with the same rules as in the continuous time version of the processes. The transition operator corresponding to a time step $\delta t = 2L^{-1}$ is

$$T_0 f(\zeta) = f(\zeta) + L^{-1}\sum_x [f(\zeta + \Delta\zeta(x)\,\delta_x) - f(\zeta)] \qquad (4.3)$$

or

$$T f(\xi) = f(\xi) + L^{-1}\sum_x \mathbf{1}\{\xi + \Delta\xi(x)\,\delta_x \geqslant 0\} [f(\xi + \Delta\xi(x)\,\delta_x) - f(\xi)] \qquad (4.4)$$

$Tf(\xi)$ is the expected value of the function f evaluated at time $2L^{-1}$ (after one step) when the initial configuration is ξ for the discretized version of the process. The same interpretation is valid for $T_0 f(\zeta)$. By abuse of notation we call the discrete time versions of the process ξ_t and ζ_t as we did for the continuous time versions. Notice that

$$\lim_{L \to \infty} \frac{Tf(\xi) - f(\xi))}{\delta t} = \mathscr{L} f(\xi) \; ; \qquad \lim_{L \to \infty} \frac{T_0 f(\zeta) - f(\zeta))}{\delta t} = \mathscr{L}_0 f(\zeta) \quad (4.5)$$

The processes ξ_t and ζ_t are coupled in the simplest possible way: the same random sequence of sites are used for both. Notice however that this coupling is different from the one described in Section 3 (in particular it is not attractive in the sense that it does not necessarily satisfy (1.8) but it is faster). For L finite the discrete time and continuous time processes can be identified up to a time change, using the ordered sequence of updated sites. The random time involved in the time change has fluctuations which should be negligible for our purposes.

   The numerical samples for the data shown below were drawn using either the Mersenne Twister pseudorandom integer generator, see Matsumoto and Nishimura,[28] or the R250 pseudorandom generator, see Kirkpatrick and Stoll.[29] The length L is 10^6 or $2^{20} = 1024^2$ and time runs up to 2.10^6 or 2^{21}. The number of calls to the generator for the realization of one sample of length L up to time t is $L.t/2 \leqslant 10^{12}$, which of course is much less than the period of the generator (a necessary but not sufficient condition for reliability). We compute empirical averages

$$\overline{\xi_t^2} = L^{-1} \sum_x \xi_t(x)^2, \qquad \overline{\zeta_t^2} = L^{-1} \sum_x \zeta_t(x)^2 \quad (4.6)$$

and empirical distribution functions

$$f_t(n) = L^{-1} \sum_x \mathbf{1}\{\xi_t(x) = n\}, \quad f_{0,t}(n) = L^{-1} \sum_x \mathbf{1}\{\zeta_t(x) = n\}, \quad n \in \mathbb{Z} \quad (4.7)$$

scaled into

$$\phi_t(s) = t^{1/4} L^{-1} \sum_x \mathbf{1}\{t^{-1/4}\xi_t(x) = s\},$$

$$\phi_{0,t}(s) = t^{1/4} L^{-1} \sum_x \mathbf{1}\{t^{-1/4}\zeta_t(x) = s\}, \qquad s \in t^{-1/4}\mathbb{Z} \quad (4.8)$$

which, extended to $s \in \mathbb{R}$ approximate the Schwartz distributions

$$\tilde{\phi}_t(s) = L^{-1} \sum_x \delta(t^{-1/4}\xi_t(x) - s), \quad \tilde{\phi}_{0,t}(s) = L^{-1} \sum_x \delta(t^{-1/4}\zeta_t(x) - s), \quad s \in \mathbb{R} \quad (4.9)$$

where $L^{-1} \sum_x$ is an ersatz for the expectation over a real random variable, limit of $t^{-1/4} \xi_t(x)$.

The processes were studied for time $t \leqslant 2L$, whereas the effect of finite size with periodic boundary conditions is expected to be visible only after a time of order L^2, the relaxation time of an interface of length L. The law of large numbers in empirical averages as above is believed to be at work with an effective number of weakly dependent variables of order $L/t^{1/2}$: the interface at time t can be thought of as a collection of $L/t^{1/2}$ segments of length $t^{1/2}$, the different segments being weakly dependent. For $t = L$ and one sample of the process, we have only $t^{1/2}$ independent segments, hence an expected relative statistical error of order $t^{-1/4}$. This explains the more erratic behaviour at larger times in Fig. 1.

The numerical experiment clearly favors an asymptotic behavior $\mathbb{E} \xi_t(0)^2 \sim t^{1/2} \log t$ as $t \to \infty$.

Figure 2 shows the scaled empirical distribution functions at various large times. Clearly $t^{-1/4} \zeta_t(x)$ converges to a centered Gaussian random
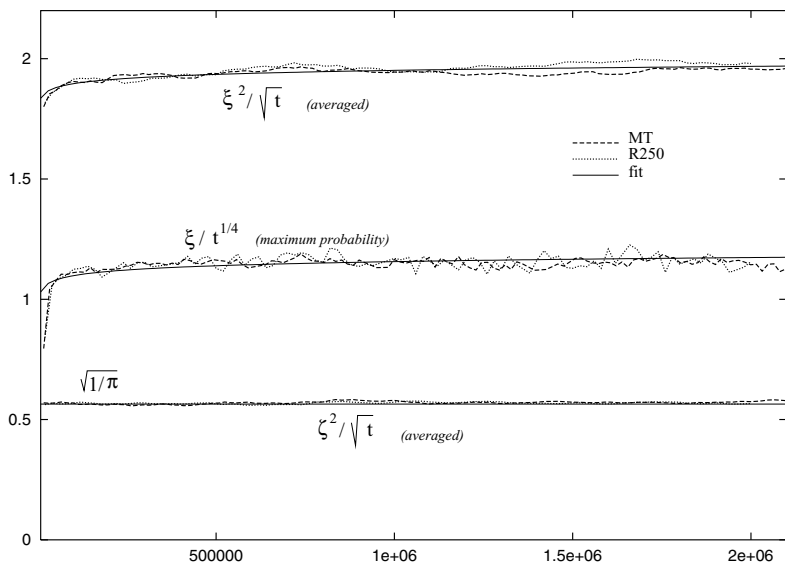

Fig. 1. From top to bottom, as function of time: $t^{-1/2} \overline{\xi_t^2}$ together with best fit $1.62 + 0.024 \log t$; the value of s where $\phi_t(s)$ is maximum, together with best fit $\sqrt{0.55 + 0.057 \log t}$; and $t^{-1/2} \overline{\zeta_t^2}$ together with the exact asymptotic value $\sqrt{1/\pi}$. Graphs labelled MT are averages over 6 runs with the MT random generator with different seeds. Graphs labelled R250 are averages over 5 runs with the R250 random generator with different seeds. Interface length is $L = 2^{20}$ or $L = 10^6$.
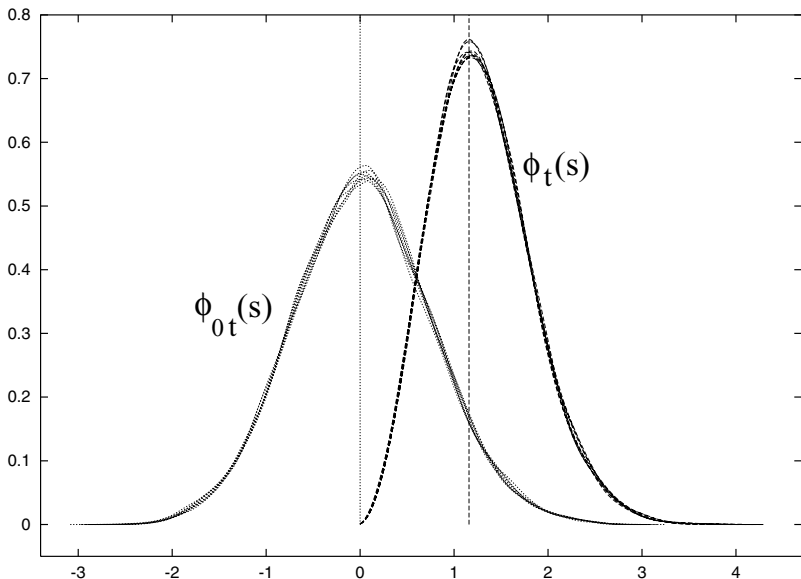
Fig. 2. $\phi_t(s)$ and $\phi_{0,t}(s)$ at large times, from the same data as in Fig. 1.

variable as expected. The distribution function of $t^{-1/4}\xi_t(x)$ is markedly asymmetrical. Zooming around $s = 0$ indicates $\phi_\infty(0) = \phi'_\infty(0) = 0$ and $\phi''_\infty(0) > 0$, and up to a possible logarithmic correction

$$f_t(0) = L^{-1} \sum_x \mathbf{1}\{\xi_t(x) = 0\} \sim t^{-3/4} \tag{4.10}$$

5. INTERFACE OF THE ISING MODEL AT ZERO TEMPERATURE

In this section we explain the relation of our model with the interface of a particular Ising model at zero temperature. Let the "inverse temperature" $\beta \geqslant 0$ and $\sigma_t \in \{-1, +1\}^{\mathbb{Z}^2}$ be the Ising model with generator

$$\mathscr{L}_\beta f(\sigma) = \tfrac{1}{2} \sum_{x \in \mathbb{Z}^2} c_\beta(x, \sigma) \left[f(\sigma^x) - f(\sigma) \right]$$

with $\sigma^x(z) = \sigma(z)$ for $z \neq x \in \mathbb{Z}^2$, $\sigma^x(x) = -\sigma(x)$ and $c_\beta(x, \sigma)$ are the Glauber rates

$$c_\beta(x, \sigma) = \frac{e^{-\beta H(\sigma^x)}}{e^{-\beta H(\sigma^x)} + e^{-\beta H(\sigma)}}$$

with Hamiltonian

$$H(\sigma) = -\sum_x \sum_{y\,:\,|y-x|=1} \sigma(x)\,\sigma(y) - h \sum_{x\,:\,x_1 > x_2} \sigma(x)$$

for some positive magnetic field $h > 0$. Consider the case $\beta = \infty$ and assume that the starting configuration σ_0 is "all ones" below the diagonal and "all minus ones" above or in the diagonal:

$$\sigma_0(x_1, x_2) = \begin{cases} +1, & \text{if} \quad x_1 > x_2; \\ -1, & \text{if} \quad x_1 \leqslant x_2 \end{cases} \tag{5.1}$$

In this case for all t the configuration σ_t has the property that all sites have either exactly two or no neighbor with opposite sign; furthermore, only sites above or in the diagonal may be negative. As a consequence, for all $t \geqslant 0$ the rates $c(x, \sigma_t)$ are positive only for sites x above or in the diagonal for which there are exactly two neighboring sites with different sign: under initial condition (5.1),

$$c_\infty(x, \sigma_t) = \begin{cases} 1/2 & \text{if} \quad \sum_{y\,:\,|y-x|=1} \mathbf{1}\{\sigma_t(y) \neq \sigma_t(x)\} = 2 \quad \text{and} \quad x_1 \leqslant x_2 \\ 0 & \text{otherwise} \end{cases}$$

To get the wall process of Theorem 1.3 from the above dynamics with initial condition (5.1), we first rotate the lattice by $-45°$ and multiply by $\sqrt{2}$, that is, we perform the transformation $R : \mathbb{Z}^2 \to \mathbb{Z}_2^2; R(x, y) = (x+y, x-y)$, where $\mathbb{Z}_2^2 := \{(x, y) \in \mathbb{Z}^2 : x+y \text{ is even}\}$ is the even sublattice of $\mathbb{Z}^2$. The above dynamics then induces a dynamics in $\{-1, +1\}^{\mathbb{Z}_2^2}$ given by $\tilde{\sigma}_t(z) = \sigma_t(R^{-1}z)$, $z \in \mathbb{Z}_2^2$, $t \geqslant 0$. Defining $\tilde{\xi}_t(x) := \min\{y : (x, y) \in \mathbb{Z}_2^2 \text{ and } \tilde{\sigma}_t(x, y) = -1\}$, $x \in \mathbb{Z}$, $t \geqslant 0$, we have that the wall process $\xi_\cdot(\cdot)$ with generator (1.1) and initial configuration (1.4) has the same law as $\tilde{\xi}_\cdot(\cdot)$.

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